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ABSTRACT

In this article, we define the Leavitt path algebra $L_S(\Gamma)$ of a directed graph Γ with coefficients in a Clifford semifield S. The general properties of $L_S(\Gamma)$ are briefly discussed. Then, concentrating on the full k-simplicity (that is, the property of having no nontrivial full k-ideals), we find the necessary and sufficient condition for full k-simplicity of $L_S(\Gamma)$ of a directed graph Γ over a Clifford semifield S. Also, we introduce c-homomorphisms of Leavitt path algebras over Clifford semifields and establish a version of the (Cuntz-Krieger) Uniqueness theorem for the Clifford semifield setting.

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1. Introduction

Leavitt path algebra has emerged as one of the most engaging fields of study in recent times. Ever since it was introduced by Abrams and Pino in their seminal article [2], several mathematicians have extensively worked in this new topic. Having its roots in the Leavitt algebras (a class of K-algebras universal with respect to an isomorphism property between finite-rank free modules, where K is a field) introduced by Leavitt [10] in 1962, Leavitt path algebra is also significant from an analytical perspective as it connects graph C^*-algebras and Leavitt algebras. In fact, obtaining a more complete algebraic picture of the different C^*-algebras (for example, the C^*-algebra $\mathcal{O}_A$ of a finite matrix A, or the Cuntz–Krieger algebra $C^*(E)$ for a finite graph E) was a motivation behind the introduction of the Leavitt path algebra.

Abrams and Pino defined the Leavitt path algebra $L_K(E)$ of a directed graph E with coefficients in a field K. Clearly, this associates algebraic structures with graphs and, therefore, involves both graph theory and algebra. Later, Leavitt path algebras have been generalized when they were defined over rings (by M. Tomforde, cf. [15]) and over commutative semirings (by Katsov et al., cf. [9]). Abrams and Pino found that $L_K(E)$ can be realized as an algebra of the form $\mathcal{CK}_A(K)$ (the latter being the algebraic analog of $\mathcal{O}_A$), and also that the completion of $L_{\mathbb{C}}(E)$ is virtually same as $C^*(E)$. This motivated several researchers to look into the structure and properties of Leavitt path algebras in more details (cf. [1]).

In this article, we introduce the Leavitt path algebra $L_S(\Gamma)$ of a directed graph Γ with coefficients in a Clifford semifield S. Clifford semifields are a particular kind of semirings. We consider

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the general properties of the Leavitt path algebras over Clifford semifields (after discussing the basic notions regarding Clifford semifields in Section 2 and earlier results about Leavitt path algebras in Section 3). One observes that a topic of central interest of studying Leavitt path algebras so far has been to consider their simplicity or ideal-simpleness. Abrams and Pino, Katsov et al. and Tomforde all gave precedence to this aspect of Leavitt path algebras in their works. Keeping this in mind, we concentrate on the full k-simplicity of $L_S(\Gamma)$ of a directed graph Γ over a Clifford semifield S. To be precise, we find the conditions (pertaining to the properties of S and Γ) which are necessary and sufficient for $L_S(\Gamma)$ such that the latter does not possess a particular kind of ideals viz., full k-ideals (except the trivial ones). Another interesting aspect regarding the homomorphisms defined on Leavitt path algebras is the Uniqueness theorem (also known as the Cuntz–Krieger Uniqueness theorem). We introduce a special type of homomorphism called a c-homomorphism on Leavitt path algebras over Clifford semifields and establish a version of the Uniqueness theorem with regards to c-homomorphisms.

We now give some basic definitions and terminology regarding graphs.

Definition 1.1. A directed graph $\Gamma = (V, E, r, s)$ consists of two sets V and E, and two maps $r, s : E \to V$. The elements of V are called *vertices* and the elements of E are called *edges*. For any edge e in V, $s(e)$ is the *source* of e and $r(e)$ is the *range* of e. If $s(e) = v$ and $r(e) = w$, then we say that v emits e and w receives e. Informally, we can think of e as having direction from v to w. If $r(e_1) = s(e_2)$ for some edges $e_1, e_2 \in E$, we say that e_1 and e_2 are *adjacent*.

In this article, the word "graph" will denote a directed graph unless otherwise mentioned. Clearly, for any vertex v in V, $s^{-1}(v)$ is the set of all edges whose source is v, while $r^{-1}(v)$ is the set of all edges whose range is v. If v does not emit any edges (that is, if $s^{-1}(v) = \emptyset$), we call v a *sink* whereas a vertex v is called *regular* if $0<|s^{-1}(v)|<\infty$. A graph G is *row-finite* if $|s^{-1}(v)|<\infty$ for all vertices v of G. Abrams and Pino initially defined the Leavitt path algebras for row-finite graphs only. Later, they generalized the definition for any directed graph [3].

A path $p = e_1e_2 \cdots e_n$ in a graph is a sequence of edges $e_1, e_2, \ldots, e_n$ such that $r(e_i) = s(e_{i+1})$ for $i = 1, 2, \ldots, n-1$. A path is of length n if it consists of n edges. The source of p, denoted by $s(p)$, is defined to be the source of its initial edge $s(e_1)$; while (if p has finite length) the range of p, denoted by $r(p)$, is the range of its final edge $r(e_n)$. A vertex $v \in V$ is considered as being a path of length 0, with $s(v) = v = r(v)$. The set of all paths in Γ is denoted by $E^{(*)}$. A path p is called a *closed path based at* v if $s(p) = r(p) = v$. Again, a closed path based at v is called a *closed simple path at* v if $s(e_i) \neq v$ for every $i>1$. $CP(v)$ denotes the set of all closed paths based at v, and the set of all closed simple paths based at v is denoted by $CSP(v)$. A *cycle* is a closed simple path which does not visit any of its vertices (except v) more than once. Thus, a path p is a cycle if $s(p) = r(p)$ and $s(e_i) \neq s(e_j)$ for all $i \neq j$. If c is a cycle with $s(c) = r(c) = v$, then c is said to be based at v. A graph containing no cycle is called *acyclic*. Finally, an edge e is an *exit* to a cycle $p = e_1e_2 \cdots e_n$ if there exists some $i \in \{1, 2, \ldots, n\}$ such that $s(e_i) = s(e)$ but $e \neq e_i$.

2. Basic notions regarding Clifford semifields

A semiring is an algebraic system $(S, +, \cdot)$ consisting of a nonempty set S together with two binary operations "+" and "·" on S, respectively called addition and multiplication, such that $(S, +)$ and $(S, \cdot)$ are semigroups which are connected by ring-like distributivity, that is, $a(b + c) = ab + ac$, and $(a + b)c = ac + bc$ for all $a, b, c \in S$.

A *zero* of a semiring S is an element $0 \in S$ such that $a + 0 = 0 + a = a$ and $a \cdot 0 = 0 \cdot a = 0$ for all $a \in S$. An identity of a semiring S is an element $1 \in S$ such that $a \cdot 1 = 1 \cdot a = a$ for all $a \in S$. The *zeroid* of a semiring $(S, +, \cdot)$ is the set of all a in S such that $a + b = b$ or $b + a = b$ for some $b \in S$. The set of all additive idempotents of a semiring S will be denoted by $E^+(S)$. Thus $E^+(S) = \{a \in S | a + a = a\}$. Similarly, the set of all multiplicative idempotents is denoted by

$E^\bullet(S)$. A semiring S is said to be *additively* (respectively, *multiplicatively*) *idempotent* if $(S,+)$ (respectively, $(S,\cdot)$) is a band. If S is both additively and multiplicatively idempotent, then we call it an *idempotent semiring*.

A semiring S is said to be additively (respectively, multiplicatively) commutative if $(S,+)$ (respectively, $(S,\cdot)$) is a commutative semigroup. If S is both additively and multiplicatively commutative, then it is called a commutative semiring. For notations and definitions regarding semirings, we refer the reader to [5,7].

Now we discuss the motivation which led to the definition of the structure called a Clifford semifield. It is known that commutative rings with 1 which are free of ideals (that is, for which the only ideals are "trivial") are fields. In the theory of semirings various generalizations of fields are obtained by considering the semirings which are free from different types of ideals and congruences. We give the definitions of ideal and k-ideal here.

Definition 2.1. [11] Let S be a semiring.

1. A nonempty subset A of S is called an **ideal** of S if $A+A\subseteq A$, and $SA\subseteq A, AS\subseteq A$.
2. An ideal A of S is called a ***k*-ideal** of S if for any $x,y\in S, x\in A$ and either $x+y\in A$ or $y+x\in A$ imply $y\in A$ (Golan in [5] called such an ideal to be *subtractive*).
3. An ideal A of S is called a **full ideal** of S if $E^+(S)\subseteq A$.

Homomorphisms of semirings are defined in the usual way. Let S and T be two semirings. A mapping $f:S\to T$ is called a homomorphism of S into T if $f(x+y)=f(x)+f(y)$ and $f(xy)=f(x)f(y)$ for all $x,y\in S$. Let S and T be two semirings with zero. A homomorphism $f:S\to T$ is called a homomorphism of semirings with zero if it preserves the zero element, that is, $f(0)=0$. The ***kernel*** of a homomorphism f, denoted by $kerf$, is defined to be the set $kerf=\{x\in S|f(x)=0\}$. It can be shown that $kerf$ is a k-ideal for any homomorphism f (possibly the "k" in k-ideal stands for the word "kernel"). We define

$$\bar{A}=\{x\in S|x+y=z \text{ or } y+x=z \text{ for some } y,z\in A\}$$

and $\bar{A}$ is called the k-closure of A. For any ideal I of S, the smallest k-ideal containing I is called the k-closure of I. Thus, an ideal I of S is a k-ideal if and only if $\bar{I}=I$.

The semiring S itself and the zero ideal (0) (if S has a zero) are considered as *trivial* ideals of S. These ideals are also k-ideals. S is said to be ***simple*** (respectively, ***k-simple***) or simply ***ideal-free*** (respectively, ***k-ideal-free***) if the only ideals (respectively, the k-ideals) of S are the trivial ones.

In order to study the generalizations of fields, it is natural to first consider the class of semirings which are natural generalization of fields from the viewpoint of ring theory. The first such generalization is a semifield which is defined below.

Definition 2.2. An additively commutative semiring $(S,+,\cdot)$ containing 1 and satisfying $|S|\geq 2$ is called a ***semifield*** if $(S^*,\cdot)$ is a subgroup of $(S,\cdot)$.

We call a semiring $(S,+,\cdot)$ an **additive inverse semiring** if $(S,+)$ is an additive inverse semigroup, that is, for each $a\in S$, there is a unique $a'\in S$ such that $a=a+a'+a, a'=a'+a+a'$. Additive inverse semirings were first studied by Karvellas [8] in 1974. For an additive inverse semiring $(S,+,\cdot)$, Karvellas proved the following theorem.

Theorem 2.3. *Let S be an additive inverse semiring. Then for any $a,b\in S$ and $e\in E^+(S)$ we have (i) $(a')'=a$, (ii) $ab'=(ab)'=a'b$ (iii) $ab=a'b'$ and (iv) $e'=e$.*

The introduction of the concept of relative inverses in semigroups by A. H. Clifford (not to be confused with W. K. Clifford after whom *Clifford Algebra* is named) led to the study of completely regular semigroups. For semirings, Sen, Maity, and Shum laid the axiomatic formulation for completely regular semirings (cf. [12]). Completely regular semirings are disjoint unions of skew-rings (the latter being structures which have all the properties of rings except the additive commutativity). Also, Sen et al. (cf. [14]) characterized semirings which are distributive lattices of skew-rings and called them Clifford semirings (note that Ghosh [4] introduced Clifford semirings as strong distributive lattices of rings). In [13], Sen, Maity, and Shum introduced Clifford semifields. The definitions of Clifford semirings and Clifford semifields are given below.

Definition 2.4. A semiring S is called a **Clifford semiring** if it is an additive inverse semiring such that for every $a \in S$ its inverse a' satisfies

$$a + a' = a' + a \text{ and } a(a + a') = a + a'$$

and $E^+(S)$ is a distributive sublattice as well as a k-ideal of S.

Throughout the article, we assume that all the Clifford semirings contain 0.

Theorem 2.5. *[14, Theorems 3.2 and 3.3] Let S be a semiring. Then the following conditions are equivalent:*

i. *S is a Clifford semiring;*
ii. *S is an additive inverse semiring satisfying for all $a, b \in S, a + a' = a' + a; a(a + a') = a + a'$; $(a + a')b = b(a + a')$; and $a + (a + a')b = a$, where a' is the additive inverse of a, and if $a + d = d$ for some $d \in S$, then that implies $a + a = a$;*
iii. *S is a strong distributive lattice of skew-rings.*

Definition 2.6. Let S be a Clifford semiring with 1 such that $1 \notin E^+(S)$. A nonadditive idempotent element $a \in S$ is said to be left invertible if there exists an element $r \in S$ such that $ra + 1 + 1' = 1$. In this case, r is called a left inverse of a. Similarly, we can define a right invertible element in a Clifford semiring. An element is said to be invertible if it is left invertible as well as right invertible. If a is invertible, we say that a is a *unit* of S.

Definition 2.7. A Clifford semiring S is called a **Clifford semifield** if
(i) $1 \in S$ such that $1 \notin E^+(S)$,
(ii) S is commutative and
(iii) every nonadditive idempotent element of S is a unit.

Theorem 2.8. *A commutative Clifford semiring S with 1 is a Clifford semifield if and only if S is full ideal-simple.*

Proof. First, suppose that S is a Clifford semifield and let I be an ideal of S such that $E^+(S) \subsetneq I$. Then there exists an element $a \in I$ such that $a \notin E^+(S)$. Now for $a \in S \setminus E^+(S)$ there exists an element $r \in S$ such that $ar + 1 + 1' = 1$. Clearly, $ar \in I$ and also $1 + 1' \in E^+(S) \subset I$. Thus $1 = ar + 1 + 1' \in I$ and hence $I = S$.

Conversely, let S be a Clifford semiring which is full ideal-free. Let $a \in S$ be such that $a \notin E^+(S)$. Now $Sa + E^+(S)$ is an ideal of S such that $E^+(S) \subsetneq Sa + E^+(S)$. So $Sa + E^+(S) = S$. Hence, $1 = ra + e$ for some $r \in S$ and $e \in E^+(S)$. Then $1 = 1 + 1' + 1 = ra + e + 1' + 1 = ra + 1 + 1'$. Thus a is unit in S and consequently, S is a Clifford semifield. □

Example 2.9. Let F be a field and D be a distributive lattice with 0 and 1. Then $F \times D$ is a Clifford semifield.

We conclude the section by noting the following two important results. The first of them follows from [5, Proposition (6.45)], and the second one is a consequence of the first result.

Theorem 2.10. *Let S be a commutative Clifford semiring with identity. Then for every full k-ideal I of S, $M_n(I)$ is a full k-ideal of $M_n(S)$. On the other hand, for each full k-ideal J of $M_n(S)$ there exists a unique full k-ideal T of S such that $J = M_n(T)$.*

Theorem 2.11. *If S is a Clifford semifield, then $M_n(S)$ is full k-ideal simple for any $n \in \mathbb{N}$.*

3. Definitions and some earlier results of Leavitt path algebra

In this section, we discuss some basic concepts related to our study. We also mention some results obtained earlier (by various authors) regarding Leavitt path algebras.

Initially, Leavitt path algebras were defined with their coefficients belonging to a field [2], and then with their coefficients belonging to a commutative ring [15]. Later on, Katsov et al. [9] generalized the concept by defining Leavitt path algebras with coefficients in any commutative semiring.

Definition 3.1. ([9]) Let $\Gamma = (V, E, s, r)$ be a graph, S be a commutative semiring with 1 and 0 and E^* be the set of formal symbols $\{e^* | e \in E\}$. The Leavitt path algebra $L_S(\Gamma)$ of the graph Γ with coefficients in S is defined to be the Universal S-algebra generated by the set of generators $V \cup E \cup E^*$ (where $e \to e^*$ is a bijection between E and E^* with $r(e) = s(e^*)$ and $r(e^*) = s(e)$, and V, E, E^* are pairwise disjoint), satisfying the following relations:

(A1) $vw = \delta_{v,w}v$ for all $v, w \in V$;
(A2) $s(e)e = e = er(e), r(e)e^* = e^* = e^*s(e)$ for all $e \in E$;
(CK1) $e^*f = \delta_{e,f}r(e)$ for all $e, f \in E$;
(CK2) $v = \sum_{e \in s^{-1}(v)} ee^*$ for any regular vertex v.

Elements of the set E^* are called *ghost edges*, and elements of E are called *real edges*.

Any collection $V \cup E \cup E^*$ satisfying the conditions given in Definition 3.1 is called a Leavitt-Γ family in $L_S(\Gamma)$.

Remark 3.2. Suppose $\Gamma = (V, E, s, r)$ is a graph, S is a commutative semiring and A is an S-algebra generated by the three subsets $\{a_v | v \in V\}, \{a_e | e \in E\}, \{a_{e^*} | e^* \in E^*\}$ of A for which the following hold:

1. $a_v a_w = \delta_{v,w} a_v$ for all $v, w \in V$;
2. $a_{s(e)} a_e = a_e = a_e a_{r(e)}, a_{r(e)} a_{e^*} = a_{e^*} = a_{e^*} a_{s(e)}$ for all $e \in E, e^* \in E^*$;
3. $a_{e^*} a_f = \delta_{e,f} a_{r(e)}$ for all $e, f \in E$;
4. $a_v = \sum_{e \in s^{-1}(v)} a_e a_{e^*}$ for any regular vertex v.

Then, there always exists a unique S-algebra homomorphism $\phi : L_S(\Gamma) \to A$ given by $\phi(v) = a_v, \phi(e) = a_e, \phi(e^*) = a_{e^*}$ for all $v \in V, e \in E, e^* \in E^*$. The uniqueness of the Leavitt path algebra associated to a graph Γ and a semiring S follows from the universal property.

Remark 3.3. From the four defining relations of a Leavitt path algebra (given in Definition 3.1), one can deduce the following regarding the product of the general elements of $V \cup E \cup E^*$:

i. $ef = er(e)s(f)f = \delta_{r(e),s(f)} ef$, for any $e, f \in E$.
ii. $e^*f^* = \delta_{s(e),r(f)} e^*f^*$ for any $e^*, f^* \in E^*$.

Hence, the product of two edges e_i and e_j is nonzero if and only if e_i and e_j are adjacent in the graph Γ. Extending this to arbitrary number of edges $e_1, e_2, ..., e_n$, we can see that the product

$e_1e_2 \ldots e_n$ is nonzero if and only if $e_1e_2 \cdots e_n$ is path. Similarly, the product $e_n^*e_{n-1}^* \ldots e_1^*$ is nonzero if and only if $e_n^*e_{n-1}^* \cdots e_1^*$ is a ghost path.

From the defining relations of a Leavitt path algebra (given in Definition 3.1), one can also deduce the following:

Remark 3.4. (iii) $ve = \delta_{v,s(e)}e$ and $ev = \delta_{v,r(e)}e$ for any $v \in V, e \in E$.
(iv) $ve^* = \delta_{v,r(e)}e^*$ and $e^*v = \delta_{v,s(e)}e^*$ for any $v \in V, e \in E^*$.
Hence, $ve \neq 0$ only when $v = s(e)$; and $ev \neq 0$ only when $v = r(e)$.

Remark 3.5. For a path $p = e_1e_2 \cdots e_n, p^*$ is defined as $e_n^*e_{n-1}^* \cdots e_1^*$. It is easy to see that:

$$p^*q = \begin{cases} q' & \text{if } q = pq'; \\ r(p) & \text{if } p = q; \\ p'^* & \text{if } p = qp'; \\ 0 & \text{otherwise.} \end{cases}$$

We now recall the definition of local units. A semiring R is said to have a set of *local units* F if F is a set of idempotent elements in R such that for each finite subset $\{r_1, r_2, \ldots, r_n\}$ in R, there exists an element $f \in F$ for which $fr_if = r_i$ for all $1 \leq i \leq n$. In other words, a set of idempotents F in R is a set of local units for R if each finite subset of R is contained in a (unital) subsemiring of the form fRf for some $f \in F$. Katsov et al. gave the following important result regarding the existence of units and local units in $L_S(\Gamma)$.

Lemma 3.6. *[9, Proposition 2.5] Let $\Gamma = (V, E, s, r)$ be an arbitrary graph and S be a commutative semiring. Then $L_S(\Gamma)$ is a unital S-algebra if V is finite; and if V is infinite, the set of all finite sums of distinct elements of V is the set of local units of the S-algebra $L_S(\Gamma)$.*

In the following proposition, Katsov et al. showed that the elements of $V \cup E \cup E^*$ (for a graph Γ) are all nonzero and also gave the general form of the monomials in $L_S(\Gamma)$, where S is a commutative semiring.

Proposition 3.7. *[9, Proposition 2.4] For a commutative semiring S and a graph $\Gamma = (V, E, s, r)$, the Leavitt path algebra $L_S(\Gamma)$ has the following properties:*

i. *all elements of the set $V \cup E \cup E^*$ are nonzero;*
ii. *if a, b are distinct elements in S, then $av \neq bv$ for all $v \in V$;*
iii. *every monomial in $L_S(\Gamma)$ is of the form λpq^*, where $\lambda \in S$ and p, q are paths in Γ such that $r(p) = r(q)$.*

The following result is interesting to note.

Proposition 3.8. *Let S be a commutative semiring and $\Gamma = (V, E, s, r)$ be a graph. Let c be a cycle in Γ which has no exit. If c is based at some vertex v then*

$$vL_S(\Gamma)v = \left\{ \sum_{i=-m}^{n} k_ic^i | m, n \in \mathbb{N}_0, k_i \in S \text{ for } i = -m, \ldots, n \right\}$$

where $c^{-t} = (c^*)^t$ for all $t \in \mathbb{N}$, and $c^0 = v$.

Proof. Let $A = \{\sum_{i=-m}^{n} k_ic^i | m, n \in \mathbb{N}_0, k_i \in S$ for $i = -m, \ldots, n\}$. Now $A \subset vL_S(\Gamma)v$, as each cycle given by some power of c begins and ends at v. We note that the elements of $vL_S(\Gamma)v$ are linear combinations of the elements of form $\alpha\beta^*$, where $\alpha, \beta \in E^{(*)}, s(\alpha) = s(\beta) = v$ and $r(\alpha) = r(\beta)$.

Now as c is without exits, any path p in Γ with $s(p) = v$ must be of the form $c^n p'$ where $n \geq 0$ and p' is an initial subpath of c (clearly, if p contains any edge which is not an edge of c then that would give an exit in c). Thus $\alpha = c^m \alpha'$ and $\beta = c^n \beta'$ for some $m, n \geq 0$. As α', β' are subpaths of c and $r(\alpha') = r(\beta')$, we must have that $\alpha' = \beta'$. Let $\alpha' = e_1 e_2 ... e_k$. For any edge e of c, e is the only vertex that has $s(e)$ as its source (since c is without exits). Thus, from the condition (CK2), we have $ee^* = s(e)$ for any edge e in c. Now $\alpha'(\beta')^* = e_1 e_2 \cdots e_{k-1} e_k e_k^* e_{k-1}^* \cdots e_1^* = e_1 e_2 \cdots e_{k-1} s(e_k) e_{k-1}^* \cdots e_1^* = e_1 e_2 \cdots e_{k-1} e_{k-1}^* \cdots e_1^* = \cdots = e_1 e_1^* = v$. Thus $\alpha\beta^* = c^m(\alpha')(\beta')^* c^{-n} = c^m v c^{-n} = c^m (c^*)^n$. Now as c has no exit, it is easy to see that $cc^* = v$. So $\alpha\beta^* = c^{m-n}$. This shows that $vL_S(\Gamma)v \subset A$. Hence we have that $vL_S(\Gamma)v = \{\sum_{i=-m}^{n} k_i c^i | m, n \in \mathbb{N}_0, k_i \in S \text{ for } i = -m, ..., n\}$. □

Now we state some results which have been used for determining the condition for simplicity of $L_S(\Gamma)$ when S is a semifield. We recall that a monomial is a *path in real edges* (respectively, *path in ghost edges*) if it contains no ghost edges (respectively, no real edges). A polynomial is in only real edges (respectively, in only ghost edges) if it is a sum of real paths (respectively, ghost paths). The following result proved useful in finding the condition for simplicity of $L_S(\Gamma)$ when S is a semifield. Recall that a commutative semiring S with 1 and 0 (with $1 \neq 0$) is called a semifield if every nonzero element of S is a unit in S.

Theorem 3.9. *[9, Lemma 3.2] Let $\Gamma = (V, E, s, r)$ be a graph with the property that every cycle in Γ has an exit. If S is a semifield and $\alpha \neq 0$ is a polynomial in only real edges then there exist $a, b \in L_S(\Gamma)$ such that $a\alpha b \in V$.*

The condition for simplicity of $L_S(\Gamma)$ was considered by Katsov et al. [9]. We recall the definitions of hereditary and saturated subsets of the vertex set of a graph.

Definition 3.10. Consider a graph $\Gamma = (V, E, s, r)$. A subset $H \subseteq V$ is called **hereditary** if $s(e) \in H \Rightarrow r(e) \in H$ for all $e \in E$; and $H \subseteq V$ is called **saturated** if for any regular vertex v, $r(s^{-1}(v)) \subseteq H \Rightarrow v \in H$.

Obviously, $\emptyset$ and V are both hereditary and saturated subsets of V.

The following result was established by Katsov et al. [9, Lemma 2.6], which is a generalization of the analogous result obtained by Abrams and Pino for fields (cf. [3, Theorem 3.1]).

Lemma 3.11. *[9, Lemma 2.6] Let $\Gamma = (V, E, s, r)$ be a graph, S be a commutative semiring, and I be an ideal of $L_S(\Gamma)$. Then, $I \cap V$ is a hereditary and saturated subset of V.*

Theorem 3.12. *[9, Theorem 3.4] A Leavitt path algebra $L_S(\Gamma)$ of a graph $\Gamma = (V, E, s, r)$ with coefficients in a semifield S is simple if and only if both of the following conditions are satisfied:*

i. *The only hereditary and saturated subsets of* V *are* $\emptyset$ *and* V*;*
ii. *Every cycle in* Γ *has an exit.*

Katsov et al. in fact settled the following problem: how far can the previous theorem be extended for the commutative ground semiring S? They proved the following theorem giving a necessary and sufficient condition for the simplicity of $L_S(\Gamma)$ for a commutative semiring.

Theorem 3.13. *[9, Theorem 3.5] The Leavitt path algebra $L_S(\Gamma)$ of a graph $\Gamma = (V, E, s, r)$ with coefficients in a commutative semiring S is ideal-simple if and only if the following three conditions are satisfied:*

i. *S is a semifield;*
ii. *The only hereditary and saturated subsets of V are $\emptyset$ and V;*
iii. *Every cycle in Γ has an exit.*

4. Leavitt path algebra $L_S(\Gamma)$ over a Clifford semifield S

Now we study Leavitt path algebras over Clifford semifields. Before going into the formal definition, we consider *S-semimodules* and *S-semialgebras* for a Clifford semiring S.

Definition 4.1. Let S be a commutative Clifford semiring with 0 and 1. An ***S*-semimodule** over the Clifford semiring S is a commutative inverse monoid $(M, +)$ with 0 together with a scalar multiplication $(s, m) \mapsto sm$ from $S \times M$ to M which satisfies the identities $(s_1s_2)m = s_1(s_2m), s(m_1 + m_2) = sm_1 + sm_2, (s_1 + s_2)m = s_1m + s_2m, 1m = m, s0_M = 0_M = 0m$ for all $s, s_1, s_2 \in S$ and $m, m_1, m_2 \in M$.

Definition 4.2. Let S be a commutative Clifford semiring with 0 and 1. Then $(A, +, \cdot)$ is said to be an ***S*-semialgebra** (or simply S-algebra) over S if

(i) $(A, +, \cdot)$ is an additive inverse semiring with 0,
(ii)$(A, +)$ is an S-semimodule and
(iii) $(sa)b = s(ab) = a(sb)$ for all $s \in S$ and $a, b \in A$.

Example 4.3. Let S be a commutative Clifford semiring with 0 and 1. Then the matrix semiring $M_n(S)$ of all $n \times n$ matrices over S is an additive inverse semiring such that $E^+(M_n(S))$ is a k-ideal of $M_n(S)$.

Example 4.4. Let U be a set and S be a commutative Clifford semiring with 0 and 1. Let A be the set of all mappings $f : U \to S$. Define addition "$+$" and multiplication "$\cdot$" on A by $(f + g)(u) = f(u) + g(u)$ and $(f \cdot g)(u) = f(u)g(u)$ for all $u \in U$. Then $(A, +, \cdot)$ is a semiring. First we show that A is an additive inverse semiring. For this, let $f \in A$, and we define a function $f' : U \to S$ by $f'(u) = (f(u))'$ for all $u \in U$. Then $f' \in A$ satisfies $f + f' + f = f$ and $f' + f + f' = f'$. Let g be any element of $V^+(f)$, where $V^+(f)$ is the set of all additive inverses of f. Then $f + g + f = f$ and $g + f + g = g$. This implies that for all $u \in U$, we have $f(u) + g(u) + f(u) = f(u)$ and $g(u) + f(u) + g(u) = g(u)$, that is, $g(u) \in V^+(f(u))$ in the semiring S. Since S is an additive inverse semiring, we must have that $g(u) = (f(u))' = f'(u)$ for all $u \in U$. Therefore, $g = f'$. Hence f' is the unique additive inverse of f in A and hence A is an additive inverse semiring. For $s \in S$ and $f \in A$ we define $sf : U \to S$ by $(sf)(u) = sf(u)$ for all $u \in U$. Then $(A, +)$ is an S-semimodule. Moreover, $(sf)g = s(fg) = f(sg)$ for all $s \in S$ and $f, g \in A$. Hence A is an S-semialgebra. Now $E^+(A)$ is an ideal of A. To show that $E^+(A)$ is a k-ideal of A, let $f, f + g \in E^+(A)$. Then for all $u \in U$, we have that $f(u), f(u) + g(u) \in E^+(S)$. Since S is a Clifford semiring, $E^+(S)$ is a k-ideal of S. This implies that $g(u) \in E^+(S)$, that is, $g(u) + g(u) = g(u)$ for all $u \in U$. Therefore, $(g + g)(u) = g(u)$ for all $u \in U$, that is, $g + g = g$, that is, $g \in E^+(A)$. Consequently, A is an additive inverse semiring such that $E^+(A)$ is a k-ideal of A.

Now we introduce the Leavitt path algebra of a graph Γ over a Clifford semiring.

Definition 4.5. Let $\Gamma = (V, E, s, r)$ be a directed graph and S be a Clifford semiring. The Leavitt path algebra $L_S(\Gamma)$ of the graph Γ with coefficients in S is the S-algebra given by the set of generators $V \cup E \cup E^*$ (where $e \mapsto e^*$ gives a bijection between E and E^*, and V, E, E^* are pairwise disjoint sets) satisfying the following relations:

1. $vu = \delta_{v,u}v$ for all $v, u \in V$;
2. $s(e)e = e = er(e), r(e)e^* = e^* = e^*s(e)$ for all $e \in E$;
3. $e^*f = \delta_{e,f}r(e)$ for all $e, f \in E$;
4. $v = \sum_{e \in s^{-1}(v)} ee^*$ whenever $v \in V$ is a regular vertex.

Remark 4.6. It is easy to see that the mappings given by $v \mapsto v$, for $v \in V$, and $e \mapsto e^*, e^* \mapsto e$ for $e \in E$, produce an involution on the algebra $L_S(\Gamma)$; and for any path $p = e_1 e_2 \ldots e_n$ there exists $p^* := e_n^* e_{n-1}^* \ldots e_1^*$. We see that the Leavitt path algebra $L_S(\Gamma)$ can also be defined as the quotient of the free S-algebra $S = <v, e, e^* \mid v \in V, e \in E, e^* \in E^*>$ by the congruence "$\sim$" generated by the following ordered pairs:

i. $(vu, \delta_{v,u} v)$ for all $v, u \in V$,
ii. $(s(e)e, e), (e, er(e))$, and $(r(e)e^*, e^*), (e^*, e^*s(e))$ for all $e \in E$,
iii. $(e^*f, \delta_{e,f} r(e))$ for all $e, f \in E$,
iv. $(v, \sum_{e \in s^{-1}(v)} ee^*)$ for all regular vertices $v \in V$.

Remark 4.7. As shown in Proposition 4.8 next, all generators $\{v, e, e^* | v \in V, e \in E, e^* \in E^*\}$ of $L_S(\Gamma)$ (for any graph $\Gamma = (V, E, s, r)$) are nonadditive idempotents. Furthermore, from the above remark, it readily follows that $L_S(\Gamma)$ is, in fact, the "largest" algebra generated by the elements $\{v, e, e^* | v \in V, e \in E, e^* \in E^*\}$ satisfying the relations (1)–(4) of Definition 4.5. In other words, $L_S(\Gamma)$ has the following universal property: If A is an S-algebra generated by a family of elements $\{a_v, b_e, c_{e^*} | v \in V, e \in E, e^* \in E^*\}$ satisfying relations analogous to the relations (1)–(4) in Definition 4.5, then there always exists an S-algebra homomorphism $\phi : L_S(\Gamma) \to A$ given by $\phi(v) = a_v, \phi(e) = b_e$ and $\phi(e^*) = c_{e^*}$.

Proposition 4.8. *If Γ is a graph and S is a Clifford semiring with identity, then the Leavitt path algebra $L_S(\Gamma)$ has the following properties:*

i. *all elements of the set $\{v, e, e^* | v \in V, e \in E\}$ are nonadditive idempotents;*
ii. *if a, b are distinct elements in S, then $av \neq bv$ for all $v \in V$.*

Proof. The proof given for the case of rings in [15, Proposition 3.4], which, in fact, uses a construction similar to that of the case of fields from [6, Lemma 1.5], is based on Remark 4.7. One needs to construct an S-algebra A as mentioned in Remark 4.7 having all generators $\{a_v, b_e, c_{e^*} | v \in V, e \in E, e^* \in E^*\}$ to be nonzero. It almost does not depend on the "abelianness" used in the case of a ring. So it works in our semiring setting as well. Just for the reader's convenience, we have decided to sketch it here.

Let I be an infinite set of cardinality at least $|V \cup E|$, and let $Z := S^{(I)}$ be a free S-semimodule with the basis I, that is, Z is a direct sum of $|I|$ copies of S. For each $e \in E$, let $A_e := Z$, and for each $v \in V$, let

$$A_v \quad := \quad \begin{cases} \bigoplus_{e \in s^{-1}(v)} A_e & \text{if } \; s^{-1}(v) \neq \emptyset; \\ Z & \text{if } \; v \text{ is a sink} \end{cases}$$

Note that all A_e's and A_v's are mutually isomorphic, since each of them is the direct sum of $|I|$ copies of S. Let $A := \bigoplus_{v \in V} A_v$. For each $v \in V$ define $T_v : A_v \to A_v$ to be the identity map and extend it to a homomorphism $T_v : A \to A$ by defining T_v to be zero on $A \ominus A_v$. Also, for each $e \in E$ choose an isomorphism $T_e : A_{r(e)} \to A_e \subseteq A_{s(e)}$ and extend it to a homomorphism $T_e : A \to A$ by mapping it to zero on $A \ominus A_{r(e)}$. Finally, we define $T_{e^*} : A \to A$ by taking the isomorphism $T_e^{-1} : A_e \subseteq A_{s(e)} \to A_{r(e)}$ and extending it to a homomorphism $T_{e^*} : A \to A$ by letting T_{e^*} to be zero on $A \ominus A_e$. Now consider the subalgebra B of $\mathrm{Hom}_S(A, A)$ generated by $\{T_v, T_e, T_{e^*} | v \in V, e \in E, e^* \in E^*\}$. For each $v \in V$ we have $A_v = S \oplus M$ for some S-semimodule M. Then $T_v(1, 0) + T_v(1, 0) = (1, 0) + (1, 0) = (2, 0) \neq (1, 0) = T_v(1, 0)$. This implies that $T_v + T_v \neq T_v$ and thus T_v is not an additive idempotent. Similarly, we can prove that T_e and T_{e^*} are also nonadditive idempotent elements for each $e \in E$. Thus $\{T_v, T_e, T_{e^*} | v \in V, e \in E, e^* \in E^*\}$ is a collection of nonadditive idempotent elements satisfying the relations described in Definition 4.5.

By the universal property of $L_S(\Gamma)$, it follows that the elements of the set $\{v, e, e^* | v \in V, e \in E, e^* \in E^*\}$ are nonadditive idempotents and thus (i) is established. Next, let a, b be two distinct elements in S. We have $aT_v(1,0) = T_v(a,0) = (a,0) \neq (b,0) = T_v(b,0) = bT_v(1,0)$. So $aT_v \neq bT_v$. The universal property of $L_S(\Gamma)$ then implies that $av \neq bv$, and thus (ii) is established. □

Now we concentrate on the properties of Leavitt path algebras defined over Clifford semifields.

Proposition 4.9. *Let Γ be a graph with the property that every cycle in Γ has an exit, and let S be a Clifford semifield. If $\alpha \in L_S(\Gamma)$ is a polynomial in only real edges whose coefficients are all in $S \setminus E^+(S)$, then there exist $a, b \in L_S(\Gamma)$ such that $a\alpha b = \lambda v$ for some $\lambda \in S \setminus E^+(S)$ and $v \in V$.*

Proof. We write the polynomial α in the form $\alpha = \sum_i \lambda_i q_i$, where the q_i's are distinct real paths and $\lambda_i \in S \setminus E^+(S)$ for all i. From the set $\{q_i\}$, we choose some p such that no proper prefix path of p is contained therein. Let $v = r(p)$. Then, using [9, Remark 2.7], we get $p^*\alpha v = \lambda v + \sum_i \lambda_i p^* q_i$, where the sum is taken over all q_i's that have p as a proper prefix path and $r(q_i) = v$, so that $p^* q_i \in CP(v)$. So writing $p^*\alpha v$ as α_1, we have that $\alpha_1 = \lambda v + \sum_{i=1}^n \lambda_i p_i$; where for each i, p_i is a closed path based on v of positive length and $0 \neq \lambda \in S$. As all coefficients of α are nonadditive idempotents, λ is a nonadditive idempotent. Let some $c \in CSP(v)$ be fixed. For any $p_i \in CP(v)$, one may write $p_i = c^{n_i} p_i'$ with $n_i \in \mathbb{N}$ maximal, so that either $p_i' = v$ or p_i' is of the form $d_i p_i''$ with $d_i \in CSP(v), d_i \neq c$. In the latter case, $(c^*)^{n_i+1} p_i = c^* p_i' = c^* d_i p_i'' = 0$ by [9, Remark 2.7]. If $n = max\{n_i | i = 1, 2, ..., n\} + 1$, then $(c^*)^n p_i c^n = p_i$ if $p_i = c^{n_i}$, and $(c^*)^n p_i c^n = 0$, otherwise. So $(c^*)^n \alpha_1 c^n = \lambda v + \sum_j \lambda_j c^{n_j}$ with $n_j > 0$. In other words, if $(c^*)^n \alpha_1 c^n$ is denoted by α' then $\alpha' = \lambda v + cP(c)$ for some polynomial P. Now let $c = e_1 e_2 \cdots e_m$. By our hypothesis and [2, Lemma 2.5], there exists an exit $f \in \Gamma$ for c. Let $s(f) = s(e_j)$ with $f \neq e_j$. Let $z = e_1 e_2 e_{j-1} \cdots f e_{j+1} \cdots e_n$. Then $s(z) = v$ and $z^* c = 0$. Now $z^* \alpha' z = z^* \lambda v z + z^* cP(c) z = \lambda z^* v z = \lambda z^* s(z) z = \lambda z^* z = \lambda r(z)$. Writing $a = z^*(c^*)^n p^*$ and $b = v c^n z$, we have that $a\alpha b = \lambda r(z)$. Thus $a\alpha b = \lambda v$ for some a, b and nonadditive idempotent λ in S and some $v \in V$. □

Corollary 4.10. *Let Γ be a graph with the property that every cycle in Γ has an exit. Also, let S be a Clifford semifield. If a full k-ideal J in $L_S(\Gamma)$ contains a nonadditive idempotent polynomial α in only real edges, then J contains a vertex.*

Proof. Since α is not an additive idempotent polynomial, α must have some coefficient which is not an additive idempotent. We write $\alpha = \delta + \beta$, where δ contains those terms whose coefficients are nonadditive idempotents and β contains those terms whose coefficients are all additive idempotents. Now $\beta \in E^+(L_S(\Gamma)) \subseteq J$ (since J is a full ideal). So both $\delta + \beta (= \alpha)$ and β belong to J. As J is a full k-ideal, this implies that $\delta \in J$. Now δ is a polynomial in only real edges whose coefficients are all nonadditive idempotents. By Proposition 4.9, we then have that there exist $a, b \in L_S(\Gamma)$ such that $a\delta b = \lambda v$ for some nonadditive idempotent $\lambda \in S$ and some $v \in V$. Now as λ is a nonadditive idempotent, it is invertible in S. So there exist $r \in S$ such that $r\lambda + 1 + 1' = 1$. Hence $r\lambda v + (1 + 1')v = v$. Now $r\lambda v = ra\delta b \in J$. Also, $(1 + 1')v$, being an additive idempotent, belongs to $E^+(L_S(\Gamma))$ and hence it belongs to J. So $v = r\lambda v + (1 + 1')v$ belongs to J. Thus J contains a vertex. □

Now we consider the product of two nonadditive idempotent polynomials in $L_S(\Gamma)$. First we have the following lemma.

Lemma 4.11. *If S is a Clifford semifield, then the product of two nonadditive idempotents cannot be an additive idempotent.*

Proof. Let a, b be two nonadditive idempotent elements in S. If possible, let $ab = e$ be an additive idempotent. Now as a is a nonadditive idempotent and S is a Clifford semifield, there exists $r \in S$ such that $ra + 1 + 1' = 1$. So $rab + (1 + 1')b = b$. This implies that $b = re + (1 + 1')b$. Now re and $(1 + 1')b$ are both additive idempotents (since e and $1 + 1'$ are additive idempotents), and hence b is an additive idempotent. This is a contradiction. So ab cannot be an additive idempotent. □

Proposition 4.12. *The product of two nonadditive idempotent polynomials in $L_S(\Gamma)$ (where S is a Clifford semifield and Γ is a graph) cannot be an additive idempotent polynomial.*

Proof. Any polynomial in $L_S(\Gamma)$ is the sum of some monomials of the form λpq^*, where p, q are paths in Γ with $r(p) = r(q)$ and $\lambda \in S$. Let P_1, P_2 be two nonadditive idempotent polynomials in $L_S(\Gamma)$. So there exists at least one monomial in P_1 such that the corresponding coefficient λ_1 is a nonadditive idempotent element in S; and similarly P_2 contains a monomial such the corresponding coefficient λ_2 is a nonadditive idempotent in S. Now in P_1P_2, the terms whose coeffiecients have at least one additive idempotent (of S) will be additive idempotent monomials. However, P_1P_2 clearly contains a term having coefficient $\lambda_1\lambda_2$. By Lemma 4.11, $\lambda_1\lambda_2$ is a nonadditive idempotent in S. So P_1P_2 has a monomial whose coefficient is nonadditive idempotent in S. From the consideration of coefficients of the polynomials in $L_S(\Gamma)$, this shows that P_1P_2 is a nonadditive idempotent polynomial. So the product of two nonadditive idempotent polynomials in $L_S(\Gamma)$ cannot be an additive idempotent polynomial. □

5. Full k-Simplicity of $L_S(\Gamma)$

In this section, we consider the conditions for full k-simplicity of $L_S(\Gamma)$ of a graph Γ over a Clifford semifield S. For Leavitt path algebras defined over fields and semirings, the conditions for ideal-simpleness were studied in detail (cf. [2,9]). Here we look for Clifford semifields and graphs for which the corresponding Leavitt path algebras become full k-ideal simple.

Theorem 5.1. *Let Γ be a graph and S be a Clifford semifield. Also, let $E^+(L_S(\Gamma))$ be a k-ideal. If $x \in L_S(\Gamma)$ and $x \notin E^+(L_S(\Gamma))$, then there exists $\gamma \in E^{(*)}$ such that $x\gamma \notin E^+(L_S(\Gamma))$ and $x\gamma$ is a polynomial in only real edges.*

Proof. Let $x = \sum_{i=1}^{n} k_i p_i q_i^*$, where $k_i \in S, p_i, q_i \in E^{(*)}$. Suppose $x \notin E^+(L_S(\Gamma))$. We choose a vertex $v \in V$ such that $xv \notin E^+(L_S(\Gamma))$. It is easy to see that such a vertex v always exists as follows: since $x \notin E^+(L_S(\Gamma)), k_i p_i q_i^* \notin E^+(L_S(\Gamma))$ for some $i = 1, 2, ..., n$, and for that i we have $xv_i = k_i p_i q_i^* \notin E^+(L_S(\Gamma))$ where $v_i = s(q_i)$. Now having chosen such a v, by regrouping terms (if needed) we may write $xv = \sum_{j=1}^{m} x_j e_j^* + y$, where $e_j \in E^{(*)}$ with $s(e_j) = v, e_j \neq e_{j'}$ for $j \neq j'$ and y is a polynomial in only real edges. We assume that xv is represented as an element of minimal degree in ghost edges. We now consider two cases.

Case I: Let $xve_j \in E^+(L_S(\Gamma))$ for all $j = 1, 2, ..., m$. Now $xve_j = \sum_{j=1}^{m} x_j e_j^* e_j + ye_j$, which implies that $xve_j = x_j + ye_j = f_j$ (say). By our assumption, $f_j \in E^+(L_S(\Gamma))$ for all $j = 1, 2, ..., m$. Now $x_j + (y + y')e_j = f_j + y'e_j$. So $x_j e_j^* + (y' + y)e_j e_j^* = f_j e_j^* + y' e_j e_j^*$. Then

$$xv + (y + y') \sum_{j=1}^{m} e_j e_j^* = \sum_{j=1}^{m} x_j e_j^* + (y + y') \sum_{j=1}^{m} e_j e_j^* + y = \sum_{j=1}^{m} f_j e_j^* + \sum_{j=1}^{m} y' e_j e_j^* + y.$$

Multiplying both sides by v, we get that $xv + (y + y') \sum_{j=1}^{m} e_j e_j^* = \sum_{j=1}^{m} f_j e_j^* + \sum_{j=1}^{m} y' e_j e_j^* + yv = \sum_{j=1}^{m} f_j e_j^* + y(\sum_{j=1}^{m} e_j e_j^*)' + yv$. Now $xv \notin E^+(L_S(\Gamma)), (y + y') \sum_{j=1}^{m} e_j e_j^* \in E^+(L_S(\Gamma))$. As $E^+(L_S(\Gamma))$ is a k-ideal, this implies that $xv + (y + y') \sum_{j=1}^{m} e_j e_j^* \notin E^+(L_S(\Gamma))$. So

$\sum_{j=1}^{m} f_j e_j^* + y((\sum_{j=1}^{m} e_j e_j^*)' + v)$ does not belong to $E^+(L_S(\Gamma))$. Now $\sum_{j=1}^{m} f_j e_j^*$ is in $E^+(L_S(\Gamma))$ (as f_j is idempotent), so $y((\sum_{j=1}^{m} e_j e_j^*)' + v) \notin E^+(L_S(\Gamma))$. This shows that $y \notin E^+(L_S(\Gamma))$ and $(\sum_{j=1}^{m} e_j e_j^*)' + v \notin E^+(L_S(\Gamma))$. From the latter relation, we find that $v \neq \sum_{j=1}^{m} e_j e_j^*$. So there exists an edge $f \notin \{e_1, e_2, ..., e_m\}$ such that $s(f) = v$ (since $v = \sum_{e \in s^{-1}(v)} ee^*$). Then, we have that $xvf = \sum_{j=1}^{m} x_j e_j^* f + yf = yf$. Clearly, yf is in real edges only and $yf \neq 0$. As y, f are both nonadditive idempotents (note that every edge of Γ is a nonadditive idempotent in $L_S(\Gamma)$ and hence f is a nonadditive idempotent) and $yf \neq 0$, by Proposition 4.12 we have that yf is a nonadditive idempotent. Thus we have a $\gamma(= vf)$ such that $x\gamma$ is a nonadditive idempotent polynomial in only real edges.

Case II: Suppose $xve_j \notin E^+(L_S(\Gamma))$ for some $j = 1, 2, ..., m$. Now $xve_j = x_j + ye_j$, and the number of ghost edges in xve_j is strictly less than that of xv. If x_j is a polynomial in only real edges then we obtain the result by proceeding similarly to Case I (as x_j is a nonadditive idempotent). Otherwise we repeat the above process to reduce the number of ghost edges in each step until we are left with a polynomial in only real edges (this process will terminate since the number of ghost edges in xv is finite). □

Corollary 5.2. *Let Γ be a graph and S be a Clifford semifield such that $E^+(L_S(\Gamma))$ is a k-ideal of $L_S(\Gamma)$. If J is a nontrivial full k-ideal of $L_S(\Gamma)$, then J contains a nonadditive idempotent polynomial in only real edges.*

Proof. As J is a nontrivial full k-ideal, $E^+(L_S(\Gamma)) \subsetneq J$. Thus J contains a nonadditive idempotent x. Thus, from Theorem 5.1, it follows that J contains a nonadditive idempotent polynomial $x\gamma$ (for some γ) in only real edges. □

Theorem 5.3. *Let $\Gamma = (V, E, s, r)$ be a graph such that every cycle in Γ has an exit. Let S be a Clifford semifield. If J is a nontrivial full k-ideal of $L_S(\Gamma)$, then $J \cap V \neq \emptyset$ and $J \cap V$ is a hereditary and saturated subset of V.*

Proof. From Corollary 5.2, J contains a nonadditive idempotent polynomial in only real edges. So by Corollary 4.10, $J \cap V \neq \emptyset$. Also, $J \cap V$ is a hereditary and saturated subset of V (cf. Lemma 3.11). □

Theorem 5.4. *Let $\Gamma = (V, E, s, r)$ be a graph. Let S be a Clifford semifield such that $E^+(L_S(\Gamma))$ is a k-ideal of $L_S(\Gamma)$. Then $L_S(\Gamma)$ is full k-ideal simple if and only if both the following conditions are satisfied.*

i. *The only hereditary and saturated subsets of V are $\emptyset$ and V.*
ii. *Every cycle in Γ has an exit.*

Proof. First, let the conditions (i) and (ii) hold. If possible, let J be a nontrivial full k-ideal of $L_S(\Gamma)$. Then from Theorem 5.3, we have that $J \cap V$ is a nonempty hereditary saturated subset of V. By condition (i), we have that $J \cap V = V$, that is, $V \subset J$. So J contains local units and consequently, $J = L_S(\Gamma)$. This shows that $L_S(\Gamma)$ has no nontrivial full k-ideal.

Conversely, let $L_S(\Gamma)$ be free of nontrivial full k-ideals. By following the proof of [2, Theorem 3.11] (which does not use the additive inverse of the ring setting and can therefore be used for semirings also), we can show that $\emptyset$ and V are the only hereditary and saturated subsets of V. Finally, let there exist some cycle c (based on some vertex v) in Γ such that c has no exit. We show that $I = \overline{<v + c> + E^+(L_S(\Gamma))}$, which is the k-closure of $<v + c> + E^+(L_S(\Gamma))$, is a nontrivial full k-ideal of $L_S(\Gamma)$. Clearly, $E^+(L_S(\Gamma))$ is properly contained in I. We show that $v \notin I$. If possible, let $v \in I$. Then there exist monic monomials $\alpha_i, \beta_i, \gamma_j, \delta_j$ such that

$$v + \sum_{i=1}^{n} k_i\alpha_i(v+c)\beta_i + f = \sum_{j=1}^{m} l_j\gamma_j(v+c)\delta_j + g \quad \cdots (***)$$

where $k_i, l_j \in S$ for $i = 1, 2, ..., m$, and $j = 1, 2, ..., n$, and $f, g \in E^+(L_S(\Gamma))$. It is easy to see that we can take $f = g$ (if $f \neq g$, we add first f and then g to both sides of $(***)$ to get an idempotent $f + g$ on both sides). Now noting that $v(v+c)v = v + c$, we can assume that (by multiplying both sides by v if necessary) $v\alpha_i v = \alpha_i$, $v\beta_i v = \beta_i$ for $i = 1, 2, ..., n$, and $v\gamma_j v = \gamma_j$ and $v\delta_j v = \delta_j$ for $j = 1, 2, ..., m$. This shows that the monomials $\alpha_i, \beta_i, \gamma_j, \delta_j$ are elements of $vL_S(\Gamma)v$. By Proposition 3.8, it then follows that (noting that $v = c^0$ and also that $(v+c)$ commutes with c and c^*) both $\sum_{i=1}^{n} k_i\alpha_i(v+c)\beta_i$ and $\sum_{j=1}^{m} l_j\gamma_j(v+c)\delta_j$ are products of some polynomial in c, c^* with $(v+c)$. So we have that

$$v + (v+c)P(c, c^*) + f = (v+c)Q(c, c^*) + f \quad (****)$$

(unless $vfv = 0$ in which case the term f vanishes from both sides and we take $f = 0$ in that case). Writing $(c^*)^n = c^{-n}$ for any $n \in \mathbb{N}$, let $Q(c, c^*) = b_{-m}c^{-m} + \cdots + b_{-1}c^{-1} + b_0 v + b_1 c^1 + \cdots + b_y c^y$ and $P(c, c^*) = a_{-n}c^{-n} + \cdots + a_{-1}c^{-1} + a_0 v + a_1 c^1 + \cdots + a_t c^t$, where $a_i \neq 0$ for $i = -n, ..., t$ and $b_i \neq 0$ for $i = -m, ..., y$. From $(****)$, we then have

$$\begin{aligned} a_{-n}c^{-n} &+ \left(\sum_{i=-n+1}^{-1} (a_{i-1} + a_i)c^i\right) + (1 + a_{-1} + a_0 + f)v + \left(\sum_{i=1}^{t}(a_{i-1} + a_i)c^i\right) + a_t c^{t+1} \\ &= b_{-m}c^{-m} + \left(\sum_{i=-m+1}^{-1} (b_{i-1} + b_i)c^i\right) + (b_{-1} + b_0 + f)v + \left(\sum_{i=1}^{y}(b_{i-1} + b_i)c^i\right) + b_y c^{y+1} \end{aligned}$$

Comparing both sides, we have that $m = n$ and $t = y$. Then, comparing the coefficients of the negative powers of c, we have that

$$\begin{aligned} a_{-n} &= b_{-n} \quad \cdots (** -n **) \\ a_{-n} + a_{-n+1} &= b_{-n} + b_{-n+1} \quad \cdots (** -n+1 **) \\ a_{-n+1} + a_{-n+2} &= b_{-n+1} + b_{-n+2} \quad \cdots (** -n+2 **) \\ a_{-n+2} + a_{-n+3} &= b_{-n+2} + b_{-n+3} \quad \cdots (** -n+3 **) \\ &\vdots \\ a_{-3} + a_{-2} &= b_{-3} + b_{-2} \quad \cdots (** -2 **) \\ a_{-2} + a_{-1} &= b_{-2} + b_{-1} \quad \cdots (** -1 **) \end{aligned}$$

Now putting $a_{-n} = b_{-n}$ in $(** -n+1 **)$, we have that $a_{-n} + a_{-n+1} = a_{-n} + b_{-n+1}$. Next, adding a_{-n} to both sides of $(** -n+2 **)$, we have that $a_{-n} + a_{-n+1} + a_{-n+2} = a_{-n} + b_{-n+1} + b_{-n+2}$. Hence we have that $x_{-n} + a_{-n+2} = x_{-n} + b_{-n+2}$ where $x_{-n} = a_{-n} + a_{-n+1}$. Adding x_{-n} to both sides of $(** -n+3 **)$ we have that $x_{-n+1} + a_{-n+3} = x_{-n+1} + b_{-n+3}$ where $x_{-n+1} = x_{-n} + a_{-n+2}$. Proceeding this way, we obtain that $x_{-4} + a_{-2} = x_{-4} + b_{-2}$ from $(** -2 **)$. So adding x_{-4} to both sides of $(** -1 **)$, we get that $a + a_{-1} = a + b_{-1}$, where $a = x_{-4} + a_{-2}$ (which is in fact equal to $a_{-n} + a_{-n+1} + \cdots + a_{-2}$).

Now comparing the coefficients of the positive powers of c in $(****)$, we have that

$$\begin{aligned} a_t &= b_t \quad \cdots (** t **) \\ a_t + a_{t-1} &= b_t + b_{t-1} \quad \cdots (** t{-}1 **) \\ a_{t-1} + a_{t-2} &= b_{t-1} + b_{t-2} \quad \cdots (** t{-}2 **) \\ &\vdots \\ a_1 + a_0 &= b_1 + b_0 \quad \cdots (** 0 **) \end{aligned}$$

$$\bullet^{v_1} \xrightarrow{e_1} \bullet^{v_2} \xrightarrow{e_2} \bullet^{v_3} \cdots\cdots\cdots\cdots \bullet^{v_{n-1}} \xrightarrow{e_{n-1}} \bullet^{v_n}$$

Figure 1. Finite line graph Γ_n.

Now exactly in the same manner as we did with the earlier equations, we obtain from equations $(**t**)$ to $(**0**)$ that $b + a_0 = b + b_0$ where $b = a_t + a_{t-1} + \cdots + a_1$. Now comparing the constant terms in both sides of $(****)$, we get that

$$(1 + a_0 + a_{-1} + f)v = (b_0 + b_{-1} + f)v \qquad \cdots(*****)$$

Adding $(b + a)v$ to both sides of $(*****)$, we get that

$$\big(1 + (b + a_0) + (a + a_{-1}) + f\big)v = \big((b + b_0) + (a + b_{-1}) + f\big)v$$

Now as $a + a_{-1} = a + b_{-1}$ and $b + a_0 = b + b_0$, we have that $v + dv + f = dv + f$ (note that $fv = f$), where $d = b + a_0 + a + a_{-1}$. Clearly, $dv + f \neq 0$ (if $f \neq 0$, then $dv + f = 0$ implies that $0 = dv + f = dv + f + f = f$, which is a contradiction; again if $f = 0$, then $dv + f = 0$ implies that $dv = 0$, which is a contradiction by Proposition 4.8(ii)). So $(dv + f)'$ exists. We add $(dv + f)'$ to both sides to get that $v + h + h' = h + h'$ (where $h = dv + f$). Now $h + h' \in E^+(L_S(\Gamma))$. As $E^+(L_S(\Gamma))$ is a k-ideal, $v + h + h'$ and $h + h'$ both belonging to $E^+(L_S(\Gamma))$ implies that $v \in E^+(L_S(\Gamma))$. This is a contradiction (by Proposition 4.8(i)). So $v \notin I$ and thus I is a full k-ideal of $L_S(\Gamma)$ which is not equal to $L_S(\Gamma)$. This is a contradiction to our assumption. Hence every cycle in Γ must have an exit. This completes the proof. □

We finish this section by demonstrating the use of Theorem 5.4 in reestablishing the full k-simplicity of the Leavitt path algebra of the finite line graph Γ_n (Figure 1).

It is known that $L_S(\Gamma_n)$ is isomorphic to $M_n(S)$. By Theorem 2.11, $M_n(S)$ is a full k-simple algebra. However, this fact can also be justified by Theorem 4.6, since it is easy to check that the finite line graph satisfies conditions (i) and (ii) mentioned in Theorem 5.4.

6. Uniqueness theorem for Clifford semifields

Regarding Leavitt path algebras, an important aspect is the Uniqueness theorem (also known as the *Cuntz-Krieger Uniqueness theorem*). First, we state the Uniqueness theorems for Leavitt path algebras defined respectively over commutative rings with 1 and fields.

Theorem 6.1. *(The Cuntz-Krieger Uniqueness theorem, [15, Theorem 6.5]) Let every cycle in a graph Γ have an exit, and let R be a commutative ring with 1. If S is a ring and $f : L_R(\Gamma) \to S$ is a ring homomorphism with the property that $f(rv) \neq 0$ for all $v \in V$ and for all $r \in R - \{0\}$, then f is injective.*

Corollary 6.2. *[15, Corollary 6.6] Let Γ be a graph such that every cycle in Γ has an exit, and let K be a field. If S is a ring and $f : L_K(\Gamma) \to S$ is a ring homomorphism with the property that $f(v) \neq 0$ for all $v \in V$, then f is injective.*

In this section, we attempt to extend this theorem for $L_S(\Gamma)$ when S is a Clifford semifield. We first give the following definitions.

Definition 6.3. Let S and T be two Clifford semirings. Let $f : S \to T$ be a mapping. f is called a **c-homomorphism** if $f(a + b) = f(a) + f(b), f(ab) = f(a)f(b), f(0) = 0, f(1) = 1,$ and f maps additive idempotent elements of S into additive idempotent elements of T.

Definition 6.4. Let S and T be two Clifford semirings. Let $f : S \to T$ be a c-homomorphism. The **c-kernel** of f is the subset of S defined as: $cker(f) = \{x \in S | f(x)$ is an additive idempotent of $T\}$.

Definition 6.5. Let S and T be two Clifford semirings. Let $f : S \to T$ be a c-homomorphism. f is called **c-injective** if for any $a, b \in S, f(a) = f(b)$ implies that $a + e = b + e$ for some additive idempotent $e \in S$.

Now we give the following result for c-homomorphisms which is similar to the Cuntz-Krieger Uniqueness theorems.

Theorem 6.6. *Let $\Gamma = (V, E, r, s)$ be a graph such that every cycle in Γ has an exit. Suppose S is a Clifford semifield such that $E^+(L_S(\Gamma))$ is a k-ideal of $L_S(\Gamma)$. Let T be a Clifford semiring. If f is a c-homomorphism from $L_S(\Gamma)$ to T with the property that f(v) is not an additive idempotent for any $v \in V$, then f is c-injective.*

Proof. If possible, let $cker(f) \neq E^+(L_S(\Gamma))$. If $e \in E^+(L_S(\Gamma))$, then as f is a c-homomorphism, we have that $f(e)$ is an additive idempotent in T, that is, $e \in cker(f)$. Thus, $E^+(L_S(\Gamma)) \subset cker(f)$. So if possible, let $E^+(L_S(\Gamma))$ be a proper subset of $cker(f)$. Let $x \in cker(f) \setminus E^+(L_S(\Gamma))$. Then by Theorem 5.1, there exists some $\gamma \in E^{(*)}$ such that $x\gamma$ is a nonadditive polynomial in $cker(f)$ in only real edges. Now as $cker(f)$ is a full k-ideal, this implies that $cker(f)$ contains a vertex (by Corollary 4.10). Let v be such a vertex. Then, $f(v)$ is an idempotent, which contradicts our assumption that $f(V) \cap E^+(T) = \emptyset$. Thus $cker(f) = E^+(L_S(\Gamma))$. Let $a, b \in L_S(\Gamma)$ such $f(a) = f(b)$. Then $f(a + b') = f(a) + f(b') = f(b) + f(b') = f(b + b')$. As $b + b'$ is an additive idempotent and f is a c-homomorphism, $f(b + b')$ is an additive idempotent. So $f(a + b')$ is an additive idempotent. Hence $a + b' \in (cker(f) =)\ E^+(L_S(\Gamma))$. So there exists $d \in E^+(L_S(\Gamma))$ such that $a + b' = d$. Then $a + b' + b = d + b = b + d$. So we have found idempotents $u\ (= b + b'), v\ (= d)$ such that $a + u = b + v$. Now $u + v$, being the sum of two additive idempotents, is an additive idempotent. So, $a + u + u = b + v + u \Rightarrow a + u = b + u + v \Rightarrow a + u + v = b + u + v + v \Rightarrow a + u + v = b + u + v$. Thus $f(a) = f(b)$ implies that $a + w = b + w$ for an additive idempotent $w\ (= u + v)$. Hence, f is c-injective. □

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